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THE STABILITY OF SECOND ORDER
QUADRATIC DIFFERENTIAL EQUATIONS.
PART II,

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The Stability of Second Order

Quadratic Differential Equations

Part II

Daniel E. Koditschek and Kumpati S. Narendra

1. <u>Introduction</u>: In a previous report [1] an exhaustive discussion of the qualitative behavior of the solutions to the equation

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x^T G x \\ x^T H x \end{bmatrix} \triangleq B(x) \tag{1}$$

led to a complete classification of stable and unstable characteristics. It was shown that system (1) can never be asymptotically stable and that a necessary and sufficient condition for the origin to be stable is that B(x) be of the form

$$B(x) = c^T x D x \tag{2}$$

where the matrix D has complex eigenvalues. In this report we consider a more general differential equation of the form

$$\dot{x} = Ax + B(x) \tag{3}$$

with both linear and quadratic terms. While the qualitative behavior of system (2) was exhausted [1] using a few simple prototypes, the addition of the linear part in (3) results in much more varied behavior: a simple classification will not be complete; a complete classification will not be simple. Our main interest is consequently in the asymptotic stability in the large of equation (3). The principal result presented here is that the conditions

(i) A is a stable matrix

(ii) $B(x) = c^T x D x$

(iii) the matrix D has complex eigenvalues

(iv) the matrix $A^{-1}D$ has complex eigenvalues

(v) the matrix $A + \lambda D$ has the eigenvector c_i for some real λ and its corresponding eigenvalue, γ, is negative

are necessary and sufficient for the asymptotic stability in the large of the equilibrium state $x = 0$. In section 2, it is shown that these conditions are necessary. In section 3 the conditions are also shown to be sufficient.

2. Necessary Conditions:

By the asymptotic stability in the large of system (3) we mean that the solutions are bounded for bounded $x(0)$, and that all solutions $x(t)$ tend to the origin as $t \to \infty$. In this section we derive restrictions on equation (3) necessary for asymptotic stability in the large.

a. Local Attractiveness: The Linear Part

A well-known result of Lyapunov theory states that if the linearized equations around an equilibrium are either asymptotically stable or unstable, the corresponding nonlinear equations are also asymptotically stable or unstable. Obviously the linearized equations of (3) around the origin are

$$\dot{x} = Ax \tag{4}$$

Hence, a necessary condition for asymptotic stability is that the eigenvalues of the A matrix have non-positive real parts and that its eigenvalues on the imaginary axis be simple. Since A is a 2 x 2 matrix this only excludes having two zero eigenvalues.

b. Global Boundedness: The Quadratic Part B(x)

If the stability of the linear equation (4) determines the local properties, intuition suggests that global properties of (3) are determined by the behavior of the quadratic part i.e. $B(x)$. The stability characteristics of system (2) were discussed in detail in [1]. The following three lemmas derived from results in [1] show that this is indeed the case.

Lemma 2.1: If the linear part of system (3) is stable with A singular and the quadratic system is type (i) (i.e. $B(x) \neq 0$ at any point other than the origin) then the system has either unbounded solutions or off origin singularities.

Proof: Since A is singular, let $A = ab^T$. We consider three possible cases.

(i) Let $B(x_1) = \lambda a$ for some x_1 which does not lie on the $b_\perp$ line ($b^T b_\perp \triangleq 0$). Then the point

$$x = -\frac{1}{\lambda(b^T x_1)} x_1$$

is an off-origin singularity of equation (3).

(ii) If $B(x) \neq \lambda a$ for any $x \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ let $V = \gamma x^T a_\perp$ where $\gamma \in \mathbb{R}$ and $a^T a_\perp = 0$ $\dot{V} = \gamma B(x)^T a_\perp$ and by the assumption made on $B(x)$, $\dot{V}(x)$ is > 0 for $\gamma = 1$ or -1. Let $\dot{V}(x) > 0$ for $\gamma = +1$. Since V assumes positive values in the neighborhood of the origin the system is unstable by Lyapunov's first instability theorem.

(iii) We now consider the case where $B(x) = \lambda a$ only on $b_\perp$. Again we have two possible cases: When $a = b_\perp$, along the vector a, $Aa = 0$ and $B(a) = \lambda a$. Hence (3) behaves as a pure quadratic scalar differential equation along this line and must be unstable.

When $a \neq b_\perp$ the above argument is no longer valid but we can show that the trajectories in a sector in the neighborhood of $b_\perp$ must be directed outwards making the origin unstable. A proof of this is given in Appendix I.

Lemma 2.2: If the linear part of system (1) is stable with nonsingular matrix A and the quadratic part B(x) is of type (i) then the overall system must have off-axis singularities.

Proof: As was shown in [1] for the pure quadratic case equation (3) can be written as

$$\dot{x}_1 = a_1(v)x_1 + g(v)x_1^2$$
$$\dot{x}_2 = a_2(v)x_1 + h(v)x_1^2$$

where $v \overset{\Delta}{=} \frac{x_2}{x_1}$ $g(v)$ and $h(v)$ are second degree polynomials in v

$x^TGx = g(v)x_1{}^2$; $x^THx = h(v)x_1{}^2$ and $Ax = \begin{bmatrix} a_1(v)x_1 \\ a_2(v)x_1 \end{bmatrix}$.

v = a constant denotes a line with constant slope in R^2. Assume (without loss of generality as shown in Appendix II) that $a_{22}g_{22} - a_{12}h_{22} \neq 0$. Then the polynomial $t(v) \overset{\Delta}{=} a_2(v)g(v) - a_1(v)h(v)$ is cubic and must have a real root v_0. In [1] it was shown that system (2) cannot be type (i) unless $h(v_0) = 0$ implies $g(v_0) \neq 0$ and vice versa.

If $h(v_0) = 0$ then $a_2(v_0) = \frac{t(v_0)}{g(v_0)} = 0$; hence a_2 and h have the common factor $(v-v_0)$ and $\dot{x}_2 \equiv 0$ along the line

$$x_0(\alpha) = \begin{bmatrix} \alpha \\ v_0\alpha \end{bmatrix}.$$

Then choosing $\alpha_0 = \frac{-a_1(v_0)}{g(v_0)}$ we have $\dot{x}_1 = \dot{x}_2 = 0$ at $x(\alpha_0)$.*

Suppose $h(v_0) \neq 0$ and $g(v_0) \neq 0$ when $t(v_0) = 0$ then let

$$-\alpha_0 = \frac{a_1(v_0)}{g(v_0)} = \frac{a_2(v_0)}{h(v_0)} \text{ and } \dot{x}_1 = \dot{x}_2 = 0 \text{ at } x_0(\alpha_0).$$

Qualitatively, if A is non-singular this proof shows that for some x, Ax and $B(x)$ have the same direction so that $\dot{x} = 0$ for a suitable choice of x. The polynomial $t(v)$ is constructed such that its zero is the desired direction v.

On the basis of lemmas 2.1 and 2.2 it is clear that a necessary condition for global asymptotic stability is that (3) may be expressed in the form

$$\dot{x} = Ax + c^TxDx \tag{5}$$

The following lemma demonstrates that the system (5) cannot be globally asymptotically stable when the matrix D has real eigenvalues. In [1] it was shown this corresponds to an unstable pure quadratic system (2). In [2] it is shown that for every quad-

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* Note $\alpha_0 = 0$ iff A is singular.

ratic system in which the field vanishes on a line there exists a function $V(x) = x^T P x$ which satisfies the Chetaev conditions of instability.

Lemma 2.3: For a system given by equation (5) in which A is stable, if D has a real eigenvalue then the origin is not globally attractive.

Proof: Let $V = x^T P x$ where P is an indefinite matrix. P is chosen in such a manner that if

$$L_1 = \{x : V(x) > 0\}$$

then $L_1 \subset L_2$ where $L_2 = \{x : x^T(PD + D^T P)x > 0\}$. Further $c_\perp \notin L_1$. Thus a closed region $V \geq 0$ is embedded in an open region $\dot{V} > 0$ and the system is unstable. [For a more detailed proof the reader is referred to [2].]

c. Other Necessary Conditions: The three previous sets of conditions in this section have had rather obvious intuitive motivation. Asymptotic stability of the linear part assures (local) asymptotic stability of the nonlinear system as well. Stability of the quadratic part is required to assure the boundedness of all solutions with bounded initial conditions. The final conditions are related to both linear and quadratic parts. They are needed to ensure the absence of off-origin singularities and a well behaved interaction between the linear and quadratic parts far away from the origin.

Lemma 2.4: The system described by the equation (5) with A asymptotically stable has off-origin singularities if and only if the matrix $A^{-1}D$ has real eigenvalues.*

Proof: Equation (3) can also be expressed as

$$\dot{x} = A[I + c^T x A^{-1} D]x$$

since A is nonsingular. If $A^{-1}D$ has an eigenvalue λ and corresponding eigenvector x_λ then

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* Unless $A^{-1}D$ has an eigenvalue of multiplicity 2, whose associated eigenvector is in the $c_\perp$ direction.

$$\frac{x_\lambda}{-\lambda(c^T x_\lambda)} \text{ is an off-origin singularity.}$$

If $c^T x_\lambda = 0$ we choose the eigenvector corresponding to the other eigenvalue.*

If $A^{-1}D$ has only complex eigenvalues then no vector x_1 exists such that $c^T x A^{-1} Dx = -x$.

Since A is nonsingular this implies that there are no off-axis singularities.

In addition to precluding off-origin singularities the condition that $A^{-1}D$ have complex eigenvalues affects the interaction of the linear and quadratic parts of (5) far away from the origin. Since the quadratic part $B(x) = 0$ along the line $c_\perp$, the influence of the linear part is felt in the neighborhood of this line even as $x \to \infty$. The following lemma addresses the critical interaction of linear and quadratic parts near $c_\perp$.

<u>Lemma 2.5</u>: If the matrices $A^{-1}D$ and D have complex conjugate eigenvalues, then for every vector $x \in \mathbb{R}^2$ there exists a λ such that x is an eigenvector of the matrix $(A+\lambda D)$.

<u>Proof</u>: Since $\gamma x \neq A^{-1}Dx$ for all $\gamma \in \mathbb{R}$ we know that $\gamma Ax \neq Dx$ for all $\gamma \in R$ and all $x \in \mathbb{R}^2$. Hence Ax and Dx are linearly independent and span $\mathbb{R}^2$ and x can be expressed as a linear combination of the two.

$$\alpha Ax + \beta Dx = x$$

or

$$[A + \beta/\alpha\ D]x = \frac{1}{\alpha}x$$

Hence choosing $\lambda = \beta/\alpha$ the result follows.

Since any direction is an eigenvector for a suitable linear combination of A and D we have, in particular

$$[A + \lambda_0 D]c_\perp = \gamma c_\perp$$

---

* Unless $A^{-1}D$ has an eigenvalue of multiplicity 2, whose associated eigenvector is in the $c_\perp$ direction.

where $c_\perp$ is the unit vector which is the orthogonal complement of c. Then writing (5) as

$$\dot{x} = [A + \lambda_0 D]x + \sigma(x)Dx$$

where

$$\sigma(x) = c^T x - \lambda_0 \tag{6}$$

the system (6) behaves essentially like a linear system

$$\dot{x} = [A + \lambda_0 D]x \tag{7}$$

at points far from the origin and in the neighborhood of the $c_\perp$ direction. In particular if $x = \frac{1}{\varepsilon_1} c_\perp + \varepsilon_2 c$, then $\sigma(x) = \varepsilon_2 c^T c - \lambda_{c_\perp}$ is a constant and the field of system (6) tends towards that of (7) as $\varepsilon_1 \to 0$.

Equations (6) and (7) provide the analytic expression of our understanding that system (5) is approximately linear not merely in the neighborhood of the origin but also for large values of x on the $c_\perp$ direction where the quadratic part vanishes.

The obvious question then is whether the linear system (7) is stable. This is answered by considering the eigenvalue γ corresponding to the eigenvector $c_\perp$ of the matrix $[A + \lambda_0 D]$. The condition $\gamma < 0$ is necessary for global asymptotic stability is demonstrated by the following lemma.

Lemma 2.6: If system (7) has an eigenvalue $\gamma > 0$ corresponding to the eigenvector $c_\perp$ then system (6) has unbounded solutions.

Proof: We will construct a region which is disconnected from the origin and which is invariant: i.e. any trajectories starting on its interior must remain so for all time. By this construction we will have violated the necessary condition that a globally asymptotically stable equilibrium state be the only proper invariant set of $\mathbb{R}^2$.

Consider system (6) when $\gamma > 0$ and assume, with no loss of generality, that $\lambda_0 < 0$. Choose a point, x_1, such that $\|x_1\| \gg 1$ and $\sigma(x_1) = 0$ where

$x_1 \overset{\Delta}{=} \frac{1}{\varepsilon_1} c_\perp + \varepsilon_2 c$, and then define $x_2 \overset{\Delta}{=} \frac{1}{\varepsilon_1} c_\perp - \varepsilon_2 c$. The region in question will be the triangular wedge bounded by the lines $\{\lambda x_1 | \lambda > 0\}$ and $\{x_1 + \lambda x_2 | \lambda > 0\}$ as shown in Figure 1. To show invariance, it suffices to show that the field on the entire boundary is directed towards the interior of the region.

At the point x_1 the system looks like equation (7)

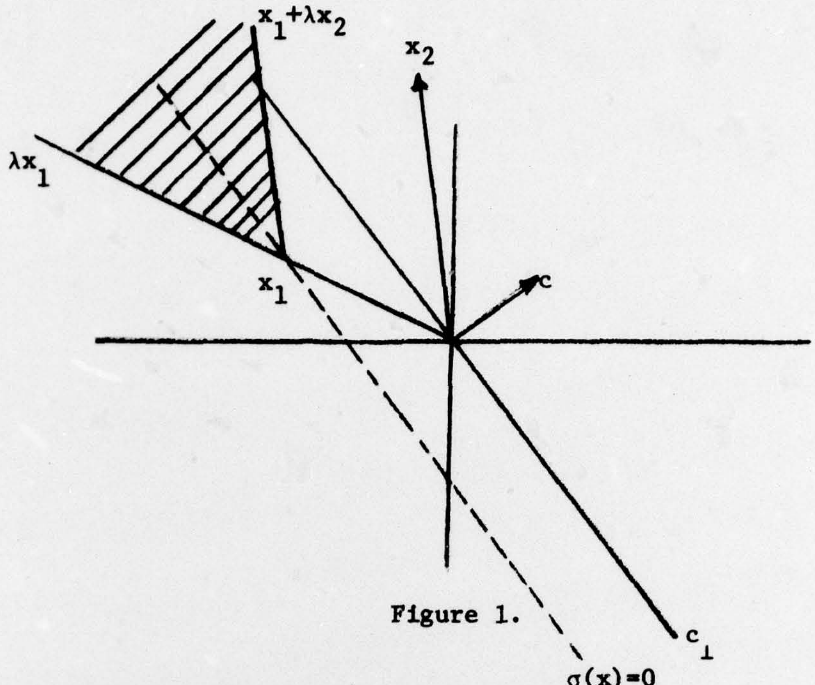


Figure 1.

$$[A - \lambda_0 D]x_1 = \gamma/\varepsilon_1 \, c_\perp + \varepsilon_3 c \ .$$

For ε_1 and ε_2 sufficiently small the field is directed toward the interior of the region. For large values of λ along either bounding line, $\sigma(x)$ becomes very large and the field is governed by $\sigma(x)Dx$. As established in [1] one half of the line through the origin in the direction $c_\perp$ is an attractive set and without loss of generality we can assume that the constructed region contains part of this attractive set. Since the pure quadratic field is directed towards the $c_\perp$ line the field in (6) is the resultant of the radially outward linear part and the tangentially inward quadratic part and is obviously directed into the interior of the region.

3. Sufficient Conditions:

In the previous section, it was shown that conditions (i)-(iv) are necessary for the asymptotic stability in the large of equation (3). In this section, it

is shown that these conditions are also sufficient.

Lemma 3.1: If in equation (6), $c_\perp$ is an eigenvector corresponding to a negative eigenvalue of the matrix $A + \lambda_0 D$, all solutions of the differential equation (3) are bounded.

Proof: This proof is based upon the behavior of the solution in three distinct regions in $\mathbb{R}^2$: a region where the quadratic part of (6) dominates the field, which we denote N; a region where the linear part of (6) dominates, denoted L; and the interface between them, denoted T. These regions will be defined with reference to three lines:

$$b \triangleq c + \varepsilon c_\perp \quad \text{(where } \varepsilon > 0 \text{ is any arbitrarily small}$$
$$d \triangleq c - \varepsilon c_\perp \qquad \text{constant of order } \frac{1}{R_0}\text{)}$$
$$\sigma(x) = c^T x - \lambda_0 \equiv 0$$

The vectors b and d define a conical region containing the line $c_\perp$ and $\sigma(x) = 0$ is an affine line parallel to $c_\perp$. Specifically, we will define

$$N \triangleq \{x \in \mathbb{R}^2 \mid x^T b d^T x \geq 0 \text{ and } \|x\| \geq R_0\}$$

as the portion of $\mathbb{R}^2$ excluding a disc of radius R_0 and the cone containing the line $c_\perp$.

$$L \triangleq \{x \in \mathbb{R}^2 \mid |\sigma(x)| \; \|D\| < \gamma_1 < \gamma, \|x\| > R_0\}$$

is an open strip around the line $\sigma(x) = 0$ upon which the linear field (7) dominates in (6).

$$T \triangleq \{x \in \mathbb{R}^2 \mid x^T b d^T x < 0, \|x\| > R_0\}$$

defines a conical region around the $c_\perp$ line and contains L for sufficiently large R_0. The sets are depicted in Figure 2.

Any point $x \in L$ lies close to the $c_\perp$ direction so that the field may be arbitrarily approximated (as R_0 increases) by

$$(A+\lambda_0 D)x + \sigma(x)Dx \approx -\gamma x \pm \frac{\gamma_1}{\|D\|} Dx$$

which is directed radially inwards. Hence, no solution which remains in L can be unbounded.

If $b_\perp^+$, $d_\perp^+$ and $b_\perp^-$ and $d_\perp^-$ denote the half line boundaries of the portion of T containing the attractive and repulsive halves of the $c_\perp$ line respectively (see Figure 2) then the field is toward the interior of N along $b_\perp^-$ and $d_\perp^-$ and toward the interior of T along $b_\perp^+$ and $d_\perp^+$. Along all trajectories in N the field is also governed by the quadratic part $c^T xDx$ with a small perturbation due to the linear part Ax. Defining $\tan\theta(x) \triangleq \frac{Ax}{\|c^T xDx\|} \leq \frac{\alpha}{\delta|c^T x|} \leq \frac{\alpha}{\delta c_1 R_0}$ (where $\alpha \triangleq \underset{\|x\|=1}{\mathrm{Max}} \|Ax\|$, $\delta = \underset{\|x\|=1}{\mathrm{Min}}$ Dx and $|c^T x| \geq c_1 R_0$ in N) we can bound the maximum angular deviation of the perturbed trajectory from that of the pure quadratic solution. Since the arc length of the unperturbed trajectory is proportional to R_0 the maximum deviation of the perturbed trajectory is bounded by

$$\int_\Gamma \frac{\alpha}{\delta c_1 R_0} ds = \alpha_0 \quad \text{(where } \Gamma \text{ is the path of the trajectory in N)}$$

where α_0 is a constant. The unperturbed trajectory leaves N at a finite point along $b_\perp^+$ and hence the perturbed solution must do so as well. Hence no solution in N can be unbounded.

Finally in T, the quadratic part predominates near the boundary and weakens as $\sigma(k) \to 0$ - i.e. as L is approached. However, near L, where the effect of the quadratic term is weak, the linear field is directed radially inward as demonstrated before. Thus, on the interior of T the field is the resultant of a tangential component and a radially directed component. Hence, no solution can become unbounded within T.

We have shown that unbounded solutions cannot occur within the set $T \cup L \cup N$. Since the set $\{x \mid \|x\| < R_0\}$ which is its complement is itself bounded this suffices to preclude the possibility of any unbounded solutions.

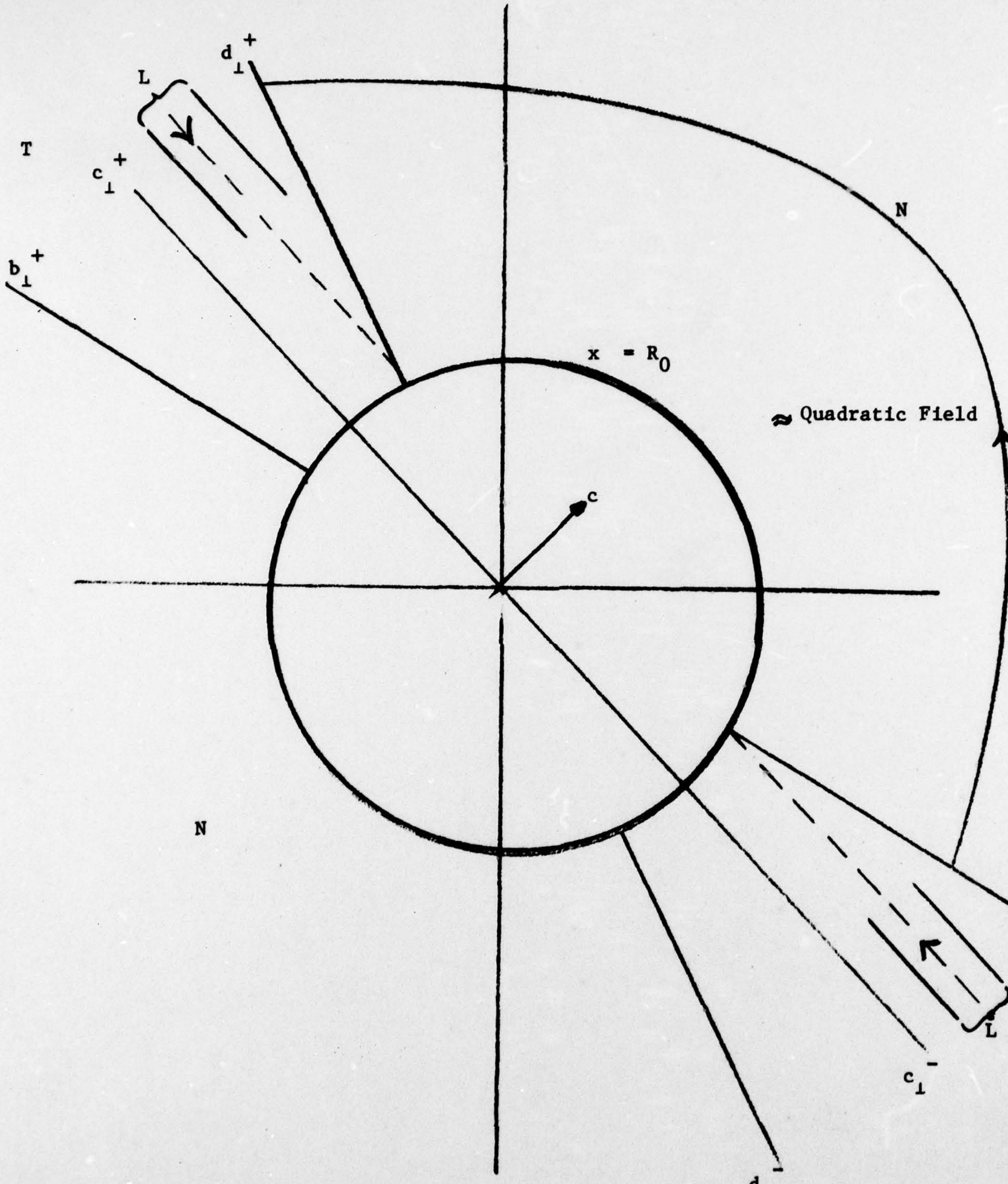
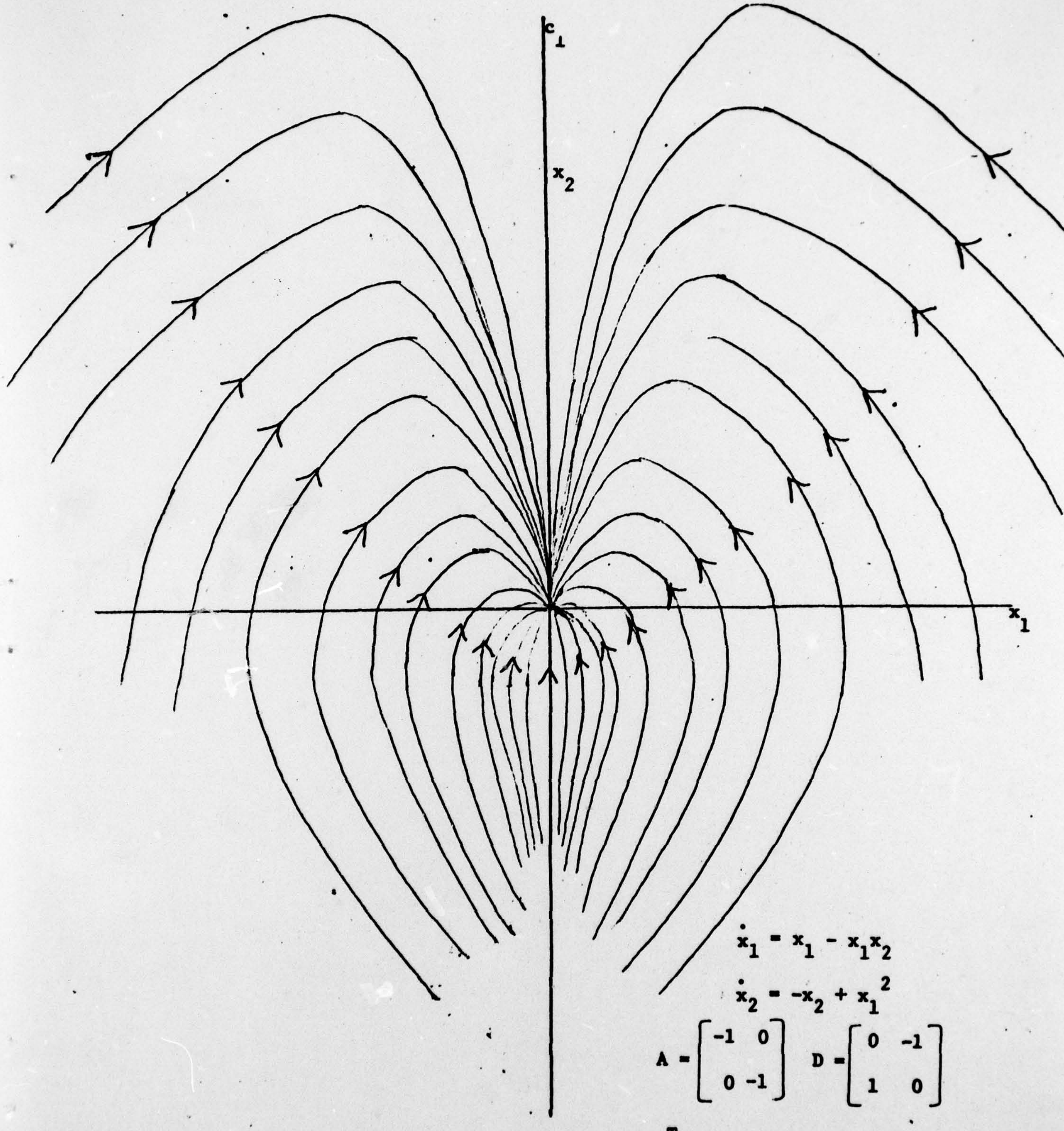


Figure 2a



$$\dot{x}_1 = x_1 - x_1 x_2$$

$$\dot{x}_2 = -x_2 + x_1^2$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$c^T = [1,0]$$

Figure 2b

Since all trajectories are bounded, the solutions must approach an invariant set. Under conditions (i)-(iv) the system has an attractive origin but no other equilibrium states and cannot sustain any limit cycles. Hence, the origin is the unique proper invariant subset in $\mathbb{R}^2$ and all solutions tend to zero. Hence, the system is globally asymptotically stable.

Conclusion:

This is a sequel to the earlier report on pure quadratic systems ~~[1]~~ and considers the complexities introduced by the addition of a linear term. The diversity of qualitative behavior evinced by such systems has led us to limit this investigation to a specific question - global asymptotic stability. Necessary and sufficient conditions for such stability are derived here taking advantage of the relatively simple structure of the pure quadratic systems discussed in ~~[1]~~. These conditions may find application in such areas of systems theory as adaptive control and bilinear systems.

Acknowledgment

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References

[1] Daniel E. Koditschek and Kumpati S. Narendra, "The Stability of Second Order Quadratic Differential Equations," S & IS Report No. 7709, Yale University, Department of Engineering and Applied Science, November, 1977.

[2] Daniel E. Koditschek and Kumpati S. Narendra, "On the Asymptotic Stability in the Large of Second Order Quadratic Differential Equations," to be published.

Appendix I

Fact: If $\dot{x} = ab^T x + B(x)$ where $B(x) = \lambda a$ if and only if $x = b_\perp$ then the origin is unstable.

Proof: In the body of the paper it was shown that the origin is unstable when $a = b_\perp$. Hence, we assume that a and $b_\perp$ are linearly independent. It can be shown that on one side of the line $b_\perp$ there is a curve C (asymptotically approaching $b_\perp$ as $\|x\|$ decreases) on which the component of B(x) in the direction of the vector 'a' is exactly equal to $(-b^T x)a$. Thus the resultant vector must be in the direction $b_\perp$ or $-b_\perp$ at every point on the entire curve C.

If vectors b_1 and b_2 are defined as

$$b_1 \stackrel{\Delta}{=} b_\perp + \varepsilon\, b$$
$$b_2 \stackrel{\Delta}{=} -b_\perp + \varepsilon\, b$$

the curve C passes through every triangular wedge bounded by b_1 and $b_\perp$ and b_2 and $-b_\perp$ as ε decreases.

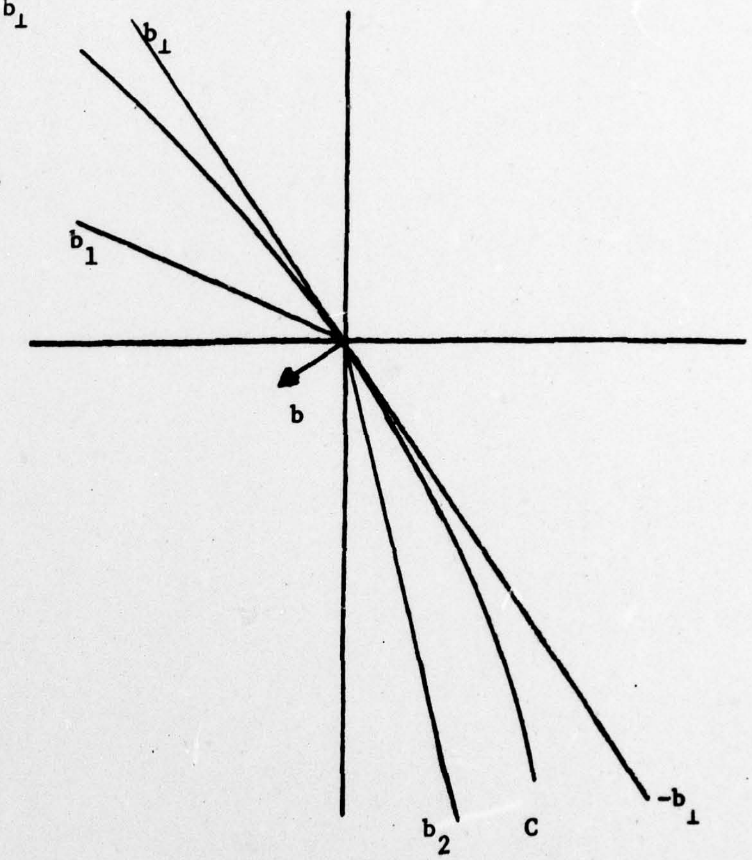

Figure 3

Further, in the neighborhood of the origin all trajectories on $b_1, b_2, b_\perp$ and $-b_\perp$ are directed towards the interior of the corresponding triangular wedge. Hence, if the component of B(x) in the $b_\perp$ direction is radially outward on one of the triangular wedges, the origin must be unstable.

It remains to show that if the resultant is radially inward on one side it must be radially outward on the other. To do this, parametrize the curve C by $c(\omega)$ where $c(0) = 0$. Since

$$\lim_{\omega\to 0} \frac{c(\omega)}{\|c(\omega)\|} = \frac{b_\perp}{\|b_\perp\|} \quad \text{and} \quad \lim_{\omega\to 0} \frac{-c(-\omega)}{\|c(-\omega)\|} = \frac{b_\perp}{\|b_\perp\|}$$

and $B(\lambda c(\omega)) = \lambda^2 B(c(\omega))$ we may study the properties of $B(x)$ over an arbitrarily small neighborhood of $b_\perp$.

Consider the mapping $v \mapsto \frac{h(v)}{g(v)}$ used in reference (1). $v \triangleq \frac{x_2}{x_1}$; $g(v) = \frac{1}{x_1^2} x^T Gx$; $h(v) = \frac{1}{x_1^2} x^T Hx$. In the vicinity of v_0 (slope of $b_\perp$) there is no change in the sign of the tangent to that curve at $h(v_0)/g(v_0)$. This implies (ref. [2]) that there exists another direction v_1 for which

$$\frac{h(v_1)}{g(v_1)} = \frac{h(v_0)}{g(v_0)}$$

But this contradicts the condition that $B(x) = \lambda a$ only on $b_\perp$. Hence $\frac{h(v)}{g(v)}$ must take on a local extremum in the neighborhood of v_0 in which case

$$b_\perp^T B(c(\omega)) < 0 \qquad \varepsilon > \omega > 0$$

and

$$-b_\perp^T B(c(-\omega)) > 0$$

so that if the resultant is radially inward on one side it is radially outward on the other and the system is unstable.

Appendix II

We will show that if $\dot{x} = Ax + B(x)$ and

$$a_{22}g_{22} - a_{12}h_{22} = 0$$

then the system has an off-origin equilibrium.

Since $\dfrac{a_{12}}{a_{22}} = \dfrac{g_{22}}{h_{22}}$, the vector $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$ and $\begin{bmatrix} g_{22} \\ h_{22} \end{bmatrix}$ are in the same direction in R^2. But

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \text{ and } B\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} g_{22} \\ h_{22} \end{bmatrix}$$

and hence

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda B\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \text{ and choosing } \gamma = -1/\lambda$$

we have

$$\dot{x} = 0 \text{ at the point } \begin{bmatrix} 0 \\ \gamma \end{bmatrix}.$$